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## **Finite Element Approach to Solution of Multidimensional Quasi- Harmonic Field Functions**

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### **Abstract**

*This paper focuses on the methodical approach for the solution of field problems whose function can be expressed as derivatives and integrated functional or on solution of quasi-harmonic functions whose physical behaviors are governed by a general quasi-harmonic differential equation that can be treated as a quadratic functional that can be minimized over a region. The functional of a stress field function was established using mixed methods analogous to variational principle, minimum total potential principle and finite element method. The functional of function,  $\Phi(x,y)$  was formed using Euler equivalent integral and finite element shape function for a function expressed in derivative form. The minimization of the functional gave the stationary values of the function which minimize the functional. The solution of the functional gave the minimum value of the function. Possible solutions of states that minimize the functional was achieved by finite element*

*solution procedure while the minimum values of the stationary states were solved by solving the functional. The functional obtained for each finite element is minimized with respect to associated degrees of freedom of the element and assembly method applied to all elements minimization equation to obtain system of equations equal to unconstrained nodes in the region. The element equations are assembled and solved by substitution to obtain the values of the function at discrete points. The values of the function at the discrete points did not vary significantly with boundary points values. The minimum value of the function representing the critical or the functional of the function is evaluated as 24MPa.*

### **Introduction**

Engineering phenomena for field problems can broadly be put into three kinds, wave phenomenon, diffusion phenomenon and potential phenomenon. While some of the complex phenomena of engineering are combination of these leading to mixed phenomenon, the three basic phenomena are modeled as, hyperbolic, parabolic and elliptic equations respectively (Sundaram et al 2003). These equations are expressed as partial differential equations that could be solved analytically for simple geometries. But partial differential equations admit infinite number of solutions when solved analytically. This is proved by the analytical solution of Laplace equation by method of separation of variables (Ihuezze 2008). For complex geometries with irregular boundaries, solution by analytical methods becomes impossible or tedious (Canale and Chapara 1998). There are many problems encountered in engineering and physics where the minimization of the integrated quantity usually referred as functional and subject to some boundary conditions results in the exact solution. This functional may represent a physical recognizable variable in some instances, for many purposes it is simply a mathematically defined entity.

Field problems and geometries are never simple so that Finite Element Method(FEM) is usually more suited for field problems solution. The FEM divides the solution domain into simply shaped regions, or elements. An approximate solution for the PDE can be developed for each of these elements. The total solution is then generated by linking together or assembling the individual solutions taking to ensure continuity at the interelement boundaries, thus the PDE is satisfied in a piecewise fashion and unique solution is obtained for a field problem.

The objective of this study is therefore to present a methodical approach to solve multiple dimensional field problems. Ihueze (2008) solved a 2-D problem, Laplace function in form by finite difference method. This same function is solved employing FEM. The functional of function;  $\Phi(x, y)$  was formed using Euler equivalent integral and finite element shape function for a function expressed in derivative form. The minimization of the functional gave the stationary values of the function which minimize the functional. The solution of the functional gave the minimum value of the function. Possible solutions of states that minimize the functional was achieved by finite element solution procedure while the minimum values of the stationary states were solved by solving the functional.

### **Theoretical Analysis**

#### **Extension of Variational Approach to Solution of Field Problems**

Quite generally, in the finite element process an approximate solution is sought to the problem of minimizing a functional. The concept of the finite element approach to elasticity as a process in which the total potential energy is minimized with respect to nodal displacements can obviously be extended to a variety of physical problems in which an extremum principle exists. In such problems the exact solution is defined as that which minimizes some integral of an unknown function or of its derivatives (Zienkiewicz 1967). This integral is known as the functional of the problem. If the unknown function is defined throughout the region, element by element in terms of the nodal values of the function, the minimization of the functional will result in a series of ordinary equations equal in number to that of the unknown values of the function at the nodes.

#### **General Field Equations and Formation of Functional of Functions**

The general quasi- harmonic equation governing the behaviour of some unknown physical quantity, had been expressed by Zienkiewicz and Cheung (1967) as

$$\frac{\partial}{\partial x} (k_x \frac{\partial \Phi}{\partial x}) + \frac{\partial}{\partial y} (k_y \frac{\partial \Phi}{\partial y}) + \frac{\partial}{\partial z} (k_z \frac{\partial \Phi}{\partial z}) + Q = 0 \quad (1)$$

in which  $\Phi$  is the unknown function assumed to be single valued within the region and  $k_x, k_y, k_z$  and  $Q$  are known specified functions of  $x, y, z$ . The well known Laplace and Poisson equations are represented respectively as

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{2}$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = f(x,y) \tag{3}$$

and are also examples of quasi – harmonic equations. The physical conditions of the particular problem and region considered will impose certain boundary conditions, where in most cases th  $\Phi$  is specified at the boundary. (1) together with the boundary conditions, specifies the extremum problem in a unique manner. However, an alternative formulation is possible with the aid of the calculus of variations. The well known Euler theorem then states that if the integral

$$I(u) = \iiint f(x,y,z,u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}) dx dy dz \tag{4}$$

is to be minimized, then the necessary and sufficient condition for this minimum to be reached is that the unknown function  $u(x,y,z)$  should satisfy the following differential equation

$$\frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial (\partial u / \partial x)} \right] + \frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial (\partial u / \partial y)} \right] + \frac{\partial}{\partial z} \left[ \frac{\partial f}{\partial (\partial u / \partial z)} \right] - \frac{\partial f}{\partial u} = 0 \tag{5}$$

within the same region. The equivalent formulation(functional) for minimization of (1) hence becomes

$x = \iiint \left\{ \frac{1}{2} [k_x \left(\frac{\partial \Phi}{\partial x}\right)^2 + k_y \left(\frac{\partial \Phi}{\partial y}\right)^2 + k_z \left(\frac{\partial \Phi}{\partial z}\right)^2] - Q \Phi \right\} dx dy dz$  subject to  $\Phi$  obeying the same boundary conditions.

**Two Dimensional FEM formulation using Triangular elements**

Physical situations in which the behaviour is essentially two- dimensional frequently arise, such as the case of Laplace or Poisson equation so that the equivalent formulation to be minimized for 2-D problem employing the condition,

$$\frac{\partial}{\partial x} \left[ \frac{\partial f}{\partial (\partial u / \partial x)} \right] + \frac{\partial}{\partial y} \left[ \frac{\partial f}{\partial (\partial u / \partial y)} \right] - \frac{\partial f}{\partial u} = 0 \tag{6}$$

becomes

$$x = \iint \left\{ \frac{1}{2} \left[ k_x \left( \frac{\partial \Phi}{\partial x} \right)^2 + k_y \left( \frac{\partial \Phi}{\partial y} \right)^2 \right] - Q \Phi \right\} dx dy \quad (7)$$

For the case of our interest, the equivalent functional to be minimized for 2-D Laplace model reduces to (8) by (7)

$$x = \iint \left\{ \frac{1}{2} \left[ k_x \left( \frac{\partial \Phi}{\partial x} \right)^2 + k_y \left( \frac{\partial \Phi}{\partial y} \right)^2 \right] \right\} dx dy \quad (8)$$

### **Methodology and Finite Element Approaches**

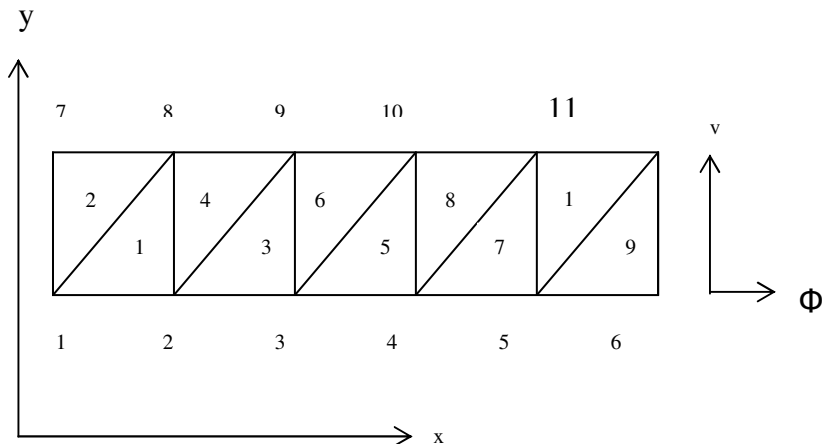
The basic steps of FEM are well developed in Astley (1992), Finlayson (1972), Zienkiewicz and Taylor (1990), Ugural and Fenster (1987), Cook et al (1989), Bathe and Wilson (1976), Hughes (1987), Zienkiewicz (1977) and Canale and Chapara (1998) and involve, discretizing, choice of approximating polynomial. Curve fitting methods and variational calculus are used with Euler approximation to obtain integrated functional to establish an equivalent functional that is minimized to obtain FE equations that will capture the response of interested function. It involves the minimization of integrated functional obtained for the region with respect to variables that minimize the response. The problem then becomes one of the constrained optimization and lends itself to approximate solution.

### **Formation of finite element shape functions and interpolation functions**

- **Discretization and Topology Definition**

This involves division of the physical system into finite sub regions to obtain a discrete model

**Fig.1 Finite Element discrete model of compressive field function of known boundary conditions.**



**Table 1 Element Topology Description**

Element number	Active degrees of freedom for assembly	Element coordinates	Element nodes
1	$\Phi_1, \Phi_2, \Phi_8, v_1, v_2, v_8$	(0,0),(17,0),(17,15)	1,2,8
2	$\Phi_1, \Phi_8, \Phi_7, v_1, v_8, v_7$	(0,0),(17,15),(0,15)	1,8,7
3	$\Phi_2, \Phi_3, \Phi_9, v_2, v_3, v_9$	(17,0),(34,0),(17,15)	2,3,9
4	$\Phi_2, \Phi_9, \Phi_8, v_2, v_9, v_8$	(17,0),(34,15),(17,15)	2,9,8
5	$\Phi_3, \Phi_4, \Phi_{10}, v_3, v_4, v_{10}$	(34,0),(51,0),(51,15)	3,4,10
6	$\Phi_3, \Phi_{10}, \Phi_9, v_3, v_{10}, v_9$	(34,0),(51,15),(34,15)	3,10,9
7	$\Phi_4, \Phi_5, \Phi_{11}, v_4, v_5, v_{11}$	(51,0),(68,0),(68,15)	4,5,11
8	$\Phi_4, \Phi_{11}, \Phi_{10}, v_4, v_{11}, v_{10}$	(51,0),(68,15),(51,15)	4,11,10
9	$\Phi_5, \Phi_6, \Phi_{12}, v_5, v_6, v_{12}$	(68,0),(85,0),(85,15)	5,6,12
10	$\Phi_5, \Phi_{12}, \Phi_{11}, v_5, v_{12}, v_{11}$	(68,0),(85,15),(68,15)	5,12,11

- **Choice of approximation function.**

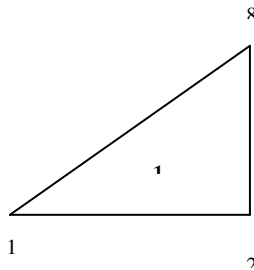
Usually polynomials are chosen and for this case linear polynomial of the form is chosen

$$\Phi(x,y) = \beta_0 + \beta_1x + \beta_2y \tag{9}$$

where  $\beta_0, \beta_1, \beta_2$  are called polynomial coefficients or shape constants.

- **Computation of shape constants**

This is achieved by curve fitting, by passing the approximating function through an element. By considering element one with degrees of freedom  $\Phi_1, \Phi_2, \Phi_8$



**Fig. 1 Element1 designations for node numbers and degrees of freedom**

$$\Phi_1 = \beta_0 + \beta_1x_1 + \beta_2y_1 \tag{10}$$

$$\Phi_2 = \beta_0 + \beta_1x_2 + \beta_2y_2 \tag{11}$$

$$\Phi_8 = \beta_0 + \beta_1x_8 + \beta_2y_8 \tag{12}$$

By putting (10) - (12) in matrix form,

$$\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_8 & y_8 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_8 \end{pmatrix} \tag{13}$$

The polynomial coefficients are solved by Cramers rule as:

$$\beta_0 = \frac{1}{A_e} [ \Phi_1(x_2y_8-x_8y_2)+ \Phi_2(x_8y_1-x_1y_8)+ \Phi_8(x_1y_2-x_2y_1)] \quad (14)$$

$$\beta_1 = \frac{1}{A_e} [ \Phi_1(y_2-y_8)+ \Phi_2(y_8-y_1)+ \Phi_8(y_1-y_2)] \quad (15)$$

$$\beta_2 = \frac{1}{A_e} [ \Phi_1(x_8-x_2)+ \Phi_2(x_1-x_8)+ \Phi_8(x_2-x_1)] \quad (16)$$

• **Derivation of shape function and interpolation functions**

$$A_e = \frac{1}{2} [ (x_2y_8-x_8y_2)+ (x_8y_1-x_1y_8)+ (x_1y_2-x_2y_1)] \quad (17)$$

By substituting (14) -(16) in (9) ,

$$\begin{aligned} \Phi(x,y) &= \frac{1}{2A_e} [ \Phi_1(x_2y_8-x_8y_2)+ \Phi_2(x_8y_1-x_1y_8)+ \Phi_8(x_1y_2-x_2y_1)] \\ &+ \frac{1}{2A_e} [ \Phi_1(y_2-y_8)x+ \Phi_2(y_8-y_1)x+ \Phi_8(y_1-y_2)x] \\ &+ \frac{1}{2A_e} [ \Phi_1(x_8-x_2)y+ \Phi_2(x_1-x_8)y+ \Phi_8(x_2-x_1)y] \\ &= \frac{\Phi_1}{2A_e} [(x_2y_8-x_8y_2)+ (y_2-y_8)x+ (x_8-x_2)y] \\ &+ \frac{\Phi_2}{2A_e} [(x_8y_1-x_8y_2)+ (y_8-y_1)x + (x_1-x_8)y] \\ &+ \frac{\Phi_8}{2A_e} [(x_1y_2-x_2y_1)+ (y_1-y_2)x+ (x_2-x_1)y] \end{aligned} \quad (18)$$

(18) can be expressed as

$$\Phi = N_1\Phi_1+ N_2 \Phi_2+ N_8 \Phi_8 \quad (19)$$



Where  $N_1, N_2, N_8$ , are shape functions at nodes 1, 2 and 8 and  $\Phi$  the interpolation function

$$N_1 = \frac{1}{2A_e} [(x_2y_8 - x_8y_2) + (y_2 - y_8)x + (x_8 - x_2)y] \quad (20)$$

$$N_2 = \frac{1}{2A_e} [(x_8y_1 - x_8y_2) + (y_8 - y_1)x + (x_1 - x_8)y] \quad (21)$$

$$N_8 = \frac{1}{2A_e} [(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] \quad (22)$$

### Formation of Functional, $X^e$ for Functions within Elements

Putting numerical values in (17), (20), (21), (22)

$$A_e = 255\text{mm}^2, N_1 = \frac{1}{510} (510 - 30x), N_2 = \frac{1}{510} (30x - 17y),$$

$$N_8 = \frac{1}{510} (17y) \quad (19)$$

then becomes

$$\Phi = \frac{\Phi_1}{510} (510 - 30x) + \frac{\Phi_2}{510} (30x - 17y) + \frac{\Phi_8}{510} (17y) \quad (23)$$

$$\frac{\partial \Phi}{\partial x} = \frac{1}{17} (-\Phi_1 + \Phi_2) \quad (24)$$

$$= \frac{1}{30} (-\Phi_2 + \Phi_8) \quad (25)$$

### Element Equations by Minimization Algorithm

This involves the formation of functional within elements and minimization of functional  $X$  expressed in (7) for all elements.

- **Element 1**

By putting (24) and (25) in (8) with  $A_e = 255\text{mm}^2$  where  $k_x = k_y = 1$

$$X^1 = 0.441 \Phi_1^2 - 0.882 \Phi_1 \Phi_2 + 0.583 \Phi_2^2 - 0.283 \Phi_2 \Phi_8 + 0.142 \Phi_8^2 \quad (26)$$

By differentiating (26) partially,

$$\left. \begin{aligned} \frac{\partial X^1}{\partial \Phi_1} &= 0.882 \Phi_1 - 0.882 \Phi_2 \\ \frac{\partial X^1}{\partial \Phi_2} &= -0.882 \Phi_2 + 1.166 \Phi_2 - 0.283 \Phi_8 \\ \frac{\partial X^1}{\partial \Phi_8} &= -0.283 \Phi_2 + 0.284 \Phi_8 \end{aligned} \right\} \quad (27)$$

• **Element 2**

By similar procedures the shape function for element 2 is derived as

$$\Phi = \frac{\Phi_1}{510} (510-17y) + \frac{\Phi_8}{510} (30x) + \frac{1}{510} (-30x+17y) \Phi_7 \quad (28)$$

and

$$\frac{\partial \Phi}{\partial x} = \frac{1}{17} (\Phi_8 - \Phi_7) \quad (29)$$

$$\frac{\partial \Phi}{\partial y} = \frac{1}{30} (-\Phi_1 + \Phi_7) \quad (30)$$

Similarly using (29) and (30) in (8) with  $A_e = 255\text{mm}^2$

$$X^2 = 0.142\Phi_1^2 - 0.283 \Phi_1 \Phi_7 + 0.441\Phi_8^2 - 0.882 \Phi_7 \Phi_8 + 0.583\Phi_7^2 \quad (31)$$

By differentiating (31) partially,

$$\left. \begin{aligned} \frac{\partial X^2}{\partial \Phi_1} &= 0.284 \Phi_1 - 0.283 \Phi_7 \\ \frac{\partial X^2}{\partial \Phi_8} &= 0.882 \Phi_8 - 0.882 \Phi_7 \\ \frac{\partial X^2}{\partial \Phi_7} &= -0.882 \Phi_8 + 1.166 \Phi_7 - 0.283 \Phi_1 \end{aligned} \right\} \quad (32)$$

By symmetry, element topology and element coordinates as specified in Fig1 and Table1 other elements equations are written as follows:

- **Element 3**

All odd numbered elements have the same symmetry and all even numbered elements have the same symmetry so that by considering **element 1** and **element 3**

$$\begin{aligned} \frac{\partial X^3}{\partial \Phi_2} &= 0.882 \Phi_2 - 0.882 \Phi_3 \\ \frac{\partial X^3}{\partial \Phi_3} &= -0.882 \Phi_2 + 1.166 \Phi_3 - 0.283 \Phi_9 \\ \frac{\partial X^3}{\partial \Phi_9} &= -0.283 \Phi_3 + 0.284 \Phi_9 \end{aligned} \tag{33}$$

- **Element 5**

All odd numbered elements have the same symmetry and all even numbered elements have the same symmetry so that by considering **element 1** and **element 5**

$$\begin{aligned} \frac{\partial X^5}{\partial \Phi_3} &= 0.882 \Phi_3 - 0.882 \Phi_4 \\ \frac{\partial X^5}{\partial \Phi_4} &= -0.882 \Phi_3 + 1.166 \Phi_4 - 0.283 \Phi_{10} \\ \frac{\partial X^5}{\partial \Phi_{10}} &= -0.283 \Phi_4 + 0.284 \Phi_{10} \end{aligned} \tag{34}$$

- **Element 7**

All odd numbered elements have the same symmetry and all even numbered elements have the same symmetry so that by considering **element 1** and **element**

$$\left. \begin{aligned}
 \frac{\partial X^7}{\partial \Phi_4} &= 0.882 \Phi_4 - 0.882 \Phi_5 \\
 \frac{\partial X^7}{\partial \Phi_5} &= -0.882 \Phi_4 + 1.166 \Phi_5 - 0.283 \Phi_{11} \\
 \frac{\partial X^7}{\partial \Phi_{11}} &= -0.283 \Phi_5 + 0.284 \Phi_{11}
 \end{aligned} \right\} \quad (35)$$

- **Element 9**

All odd numbered elements have the same symmetry and all even numbered elements have the same symmetry so that by considering **element 1** and **element 9**

$$\left. \begin{aligned}
 \frac{\partial X^9}{\partial \Phi_5} &= 0.882 \Phi_5 - 0.882 \Phi_6 \\
 \frac{\partial X^9}{\partial \Phi_6} &= -0.882 \Phi_5 + 1.166 \Phi_6 - 0.283 \Phi_{12} \\
 \frac{\partial X^9}{\partial \Phi_{12}} &= -0.283 \Phi_6 + 0.284 \Phi_{12}
 \end{aligned} \right\} \quad (36)$$

- **Element 4**

All odd numbered elements have the same symmetry and all even numbered elements have the same symmetry so that by considering **element 2** and **element 4**

$$\left. \begin{aligned}
 \frac{\partial X^4}{\partial \Phi_2} &= 0.284 \Phi_2 - 0.283 \Phi_8 \\
 \frac{\partial X^4}{\partial \Phi_9} &= 0.882 \Phi_9 - 0.882 \Phi_8 \\
 \frac{\partial X^4}{\partial \Phi_8} &= -0.882 \Phi_9 + 1.166 \Phi_8 - 0.283 \Phi_2
 \end{aligned} \right\} \quad (37)$$

- **Element 6**

All odd numbered elements have the same symmetry and all even numbered elements have the same symmetry so that by considering **element 2** and **element 6**

$$\left. \begin{aligned} \frac{\partial X^6}{\partial \Phi_3} &= 0.284 \Phi_3 - 0.283 \Phi_9 \\ \frac{\partial X^6}{\partial \Phi_{10}} &= 0.882 \Phi_{10} - 0.882 \Phi_9 \\ \frac{\partial X^9}{\partial \Phi_9} &= -0.882 \Phi_{10} + 1.166 \Phi_9 - 0.283 \Phi_3 \end{aligned} \right\} \quad (38)$$

- **Element 8**

All odd numbered elements have the same symmetry and all even numbered elements have the same symmetry so that by considering **element 2** and **element 8**

$$\left. \begin{aligned} \frac{\partial X^8}{\partial \Phi_4} &= 0.284 \Phi_4 - 0.283 \Phi_{10} \\ \frac{\partial X^8}{\partial \Phi_{11}} &= 0.882 \Phi_{11} - 0.882 \Phi_{10} \\ \frac{\partial X^8}{\partial \Phi_{10}} &= -0.882 \Phi_{11} + 1.166 \Phi_{10} - 0.283 \Phi_4 \end{aligned} \right\} \quad (39)$$

**Element 10**

All odd numbered elements have the same symmetry and all even numbered elements have the same symmetry so that by considering **element 2** and **element 10**

$$\left. \begin{aligned} \frac{\partial X^{10}}{\partial \Phi_5} &= 0.284 \Phi_5 - 0.283 \Phi_{11} \\ \frac{\partial X^{10}}{\partial \Phi_{12}} &= 0.882 \Phi_{12} - 0.882 \Phi_{11} \\ \frac{\partial X^{10}}{\partial \Phi_{11}} &= -0.882 \Phi_{12} + 1.166 \Phi_{11} - 0.283 \Phi_5 \end{aligned} \right\} \quad (40)$$

### Algorithms for Elements Assembly

This involves grouping and addition of all elements contributions to minimization,

$$\frac{\partial X}{\partial \Phi_f}$$

This will lead to system of equations equal to the **degrees of freedoms** in the continua or region. There are twelve (12) effective degrees of freedoms for elements assembling for ten elements

$$\sum \frac{\partial X^e}{\partial \Phi_f} = 0, \text{ for } f = 1, 2, 3, \dots, 12 \quad (41)$$

**For f =1**

$$\sum \frac{\partial X^e}{\partial \Phi_1} = 0 \quad (42)$$

**For f =2**

$$\sum \frac{\partial X^e}{\partial \Phi_2} = 0 \quad (43)$$

**For f =3**

$$\sum \frac{\partial X^e}{\partial \Phi_3} = 0 \quad (44)$$

**For f =4**

$$\sum \frac{\partial X^e}{\partial \Phi_4} = 0 \quad (45)$$

**For f =5**

$$\sum \frac{\partial X^e}{\partial \Phi_5} = 0 \quad (46)$$

**For f =6**

$$\sum \frac{\partial X^e}{\partial \Phi_6} = 0 \quad (47)$$

**For f =7**

$$\sum \frac{\partial X^e}{\partial \Phi_7} = 0 \quad (48)$$

**For f =8**

$$\sum \frac{\partial X^e}{\partial \Phi_8} = 0 \quad (49)$$

**For f = 9**

$$\sum \frac{\partial X^e}{\partial \Phi_9} = 0 \tag{50}$$

**For f =10**

$$\sum \frac{\partial X^e}{\partial \Phi_{10}} = 0 \tag{51}$$

**For f =11**

$$\sum \frac{\partial X^e}{\partial \Phi_{11}} = 0 \tag{52}$$

**For f =12**

$$\sum \frac{\partial X^e}{\partial \Phi_{12}} = 0 \tag{53}$$

### Assembling and formation of system of Equations

This leads to a 12x12 system of linear algebraic equations formed from (42) - (53)

**For f =1 to12 respectively**

$$\begin{aligned} \frac{\partial X}{\partial \Phi_1} &= \sum \frac{\partial X^e}{\partial \Phi_1} = 0 = \frac{\partial X^1}{\partial \Phi_1} + \frac{\partial X^2}{\partial \Phi_1} \\ &= 1.165\Phi_1 - 0.882 \Phi_2 - 0.283 \Phi_7 \end{aligned} \tag{54}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_2} &= \sum \frac{\partial X^e}{\partial \Phi_2} = 0 = \frac{\partial X^1}{\partial \Phi_2} + \frac{\partial X^3}{\partial \Phi_2} + \frac{\partial X^4}{\partial \Phi_2} \\ &= -0.882\Phi_1 + 2.331 \Phi_2 - 0.882\Phi_3 - 0.566 \Phi_8 \end{aligned} \tag{55}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_3} &= \sum \frac{\partial X^e}{\partial \Phi_3} = 0 = \frac{\partial X^3}{\partial \Phi_3} + \frac{\partial X^5}{\partial \Phi_3} + \frac{\partial X^6}{\partial \Phi_3} \\ &= -0.882\Phi_2 + 2.331 \Phi_3 - 0.882\Phi_4 - 0.566 \Phi_9 \end{aligned} \tag{56}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_4} &= \sum \frac{\partial X^e}{\partial \Phi_4} = 0 = \frac{\partial X^5}{\partial \Phi_4} + \frac{\partial X^8}{\partial \Phi_4} + \frac{\partial X^7}{\partial \Phi_4} \\ &= -0.882\Phi_3 + 2.331 \Phi_4 - 0.882\Phi_5 - 0.566 \Phi_{10} \end{aligned} \tag{57}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_5} &= \sum \frac{\partial X^c}{\partial \Phi_5} = 0 = \frac{\partial X^7}{\partial \Phi_5} + \frac{\partial X^9}{\partial \Phi_5} + \frac{\partial X^{10}}{\partial \Phi_5} \\ &= -0.882\Phi_4 + 2.331\Phi_5 - 0.882\Phi_6 - 0.566\Phi_{11} \end{aligned} \tag{58}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_6} &= \sum \frac{\partial X^c}{\partial \Phi_6} = 0 = \frac{\partial X^9}{\partial \Phi_6} \\ &= -0.882\Phi_5 + 1.166\Phi_6 - 0.283\Phi_{12} \end{aligned} \tag{59}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_7} &= \sum \frac{\partial X^c}{\partial \Phi_7} = 0 = \frac{\partial X^2}{\partial \Phi_7} \\ &= -0.283\Phi_1 + 1.166\Phi_7 - 0.882\Phi_8 \end{aligned} \tag{60}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_8} &= \sum \frac{\partial X^c}{\partial \Phi_8} = 0 = \frac{\partial X^1}{\partial \Phi_8} + \frac{\partial X^4}{\partial \Phi_8} + \frac{\partial X^2}{\partial \Phi_8} \\ &= -0.566\Phi_2 - 0.882\Phi_7 + 2.331\Phi_8 - 0.882\Phi_9 \end{aligned} \tag{61}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_9} &= \sum \frac{\partial X^c}{\partial \Phi_9} = 0 = \frac{\partial X^3}{\partial \Phi_9} + \frac{\partial X^4}{\partial \Phi_9} + \frac{\partial X^6}{\partial \Phi_9} \\ &= -0.566\Phi_6 - 0.882\Phi_8 + 2.331\Phi_9 - 0.882\Phi_{10} \end{aligned} \tag{62}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_{10}} &= \sum \frac{\partial X^c}{\partial \Phi_{10}} = 0 = \frac{\partial X^5}{\partial \Phi_{10}} + \frac{\partial X^6}{\partial \Phi_{10}} + \frac{\partial X^8}{\partial \Phi_{10}} \\ &= -0.566\Phi_4 - 0.882\Phi_9 + 2.331\Phi_{10} - 0.882\Phi_{11} \end{aligned} \tag{63}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_{11}} &= \sum \frac{\partial X^c}{\partial \Phi_{11}} = 0 = \frac{\partial X^7}{\partial \Phi_{11}} + \frac{\partial X^8}{\partial \Phi_{11}} + \frac{\partial X^{10}}{\partial \Phi_{11}} \\ &= -0.566\Phi_5 - 0.882\Phi_{10} + 2.331\Phi_{11} - 0.882\Phi_{12} \end{aligned} \tag{64}$$

$$\begin{aligned} \frac{\partial X}{\partial \Phi_{12}} &= \sum \frac{\partial X^c}{\partial \Phi_{12}} = 0 = \frac{\partial X^9}{\partial \Phi_{12}} + \frac{\partial X^{10}}{\partial \Phi_{12}} \\ &= -0.283\Phi_6 - 0.882\Phi_{11} + 1.166\Phi_{12} \end{aligned} \tag{65}$$



**Assembly Equations**

These are derived from (54)-(65) as

$$\begin{aligned}
 1.165\Phi_1 - 0.882\Phi_2 - 0.283\Phi_7 &= 0 & (66) \\
 -0.882\Phi_1 + 2.331\Phi_2 - 0.882\Phi_3 - 0.566\Phi_8 &= 0 & (67) \\
 -0.882\Phi_2 + 2.331\Phi_3 - 0.882\Phi_4 - 0.566\Phi_9 &= 0 & (68) \\
 -0.882\Phi_3 + 2.331\Phi_4 - 0.882\Phi_5 - 0.566\Phi_{10} &= 0 & (69) \\
 -0.882\Phi_4 + 2.331\Phi_5 - 0.882\Phi_6 - 0.566\Phi_{11} &= 0 & (70) \\
 -0.882\Phi_5 + 1.166\Phi_6 - 0.283\Phi_{12} &= 0 & (71) \\
 -0.283\Phi_1 + 1.166\Phi_7 - 0.882\Phi_8 &= 0 & (72) \\
 -0.566\Phi_2 - 0.882\Phi_7 + 2.331\Phi_8 - 0.882\Phi_9 &= 0 & (73) \\
 -0.566\Phi_3 - 0.882\Phi_8 + 2.331\Phi_9 - 0.882\Phi_{10} &= 0 & (62) \\
 -0.566\Phi_4 - 0.882\Phi_9 + 2.331\Phi_{10} - 0.882\Phi_{11} &= 0 & (63) \\
 -0.566\Phi_5 - 0.882\Phi_{10} + 2.331\Phi_{11} - 0.882\Phi_{12} &= 0 & (74) \\
 -0.283\Phi_6 - 0.882\Phi_{11} + 1.166\Phi_{12} &= 0 & (75)
 \end{aligned}$$

**Application of Boundary Conditions**

By applying the following boundary conditions to the system formed by (66)- (75)

$\Phi(0,0) = 154, \Phi(85,0) = 154, \Phi(0,30) = 154, \Phi(85,30) = 154$  ,the final assembly equations becomes

$$\begin{aligned}
 -0.882\Phi_2 &= -135.982 & (76) \\
 2.331\Phi_2 - 0.566\Phi_8 &= 135.828 & (77) \\
 -0.882\Phi_2 + 2.331\Phi_3 - 0.882\Phi_4 - 0.566\Phi_9 &= 0 & (78) \\
 -0.882\Phi_3 + 2.331\Phi_4 - 0.882\Phi_5 - 0.566\Phi_{10} &= 0 & (79) \\
 -0.882\Phi_4 + 2.331\Phi_5 - 0.882\Phi_6 - 0.566\Phi_{11} &= 135.828 & (80) \\
 -0.882\Phi_5 &= -135.982 & (81) \\
 -0.882\Phi_8 &= -135.982 & (82) \\
 -0.566\Phi_2 + 2.331\Phi_8 - 0.882\Phi_9 &= 135.828 & (83) \\
 -0.566\Phi_3 - 0.882\Phi_8 + 2.331\Phi_9 - 0.882\Phi_{10} &= 0 & (84) \\
 -0.566\Phi_4 - 0.882\Phi_9 + 2.331\Phi_{10} - 0.882\Phi_{11} &= 0 & (85) \\
 -0.566\Phi_5 - 0.882\Phi_{10} + 2.331 &= 135.828 & (86) \\
 -0.882\Phi_{11} &= -135.982 & (87)
 \end{aligned}$$

**Solution and Postprocessing**

The solution of the system formed by (76) - (87) is achieved by repeated substitution to obtain

$$\begin{aligned}
 \Phi_2 = 154.17, \Phi_3 = 154.68, \Phi_4 = 154.68, \Phi_5 = 154.17, \Phi_8 = 154.17, \Phi_9 \\
 = 154.68, \Phi_{10} = 154.68, \Phi_{11} = 154.17.
 \end{aligned}$$

These values are the stationary values that minimize the functional,  $X$  of the function,  $\Phi$ .

### **Computations of Elements Confirmatory Parameters**

The element parameters considered are the interelement slopes, the element integrated functionals and the derivatives of functionals considering the values of the function at their nodes. Elements symmetry are considered so that odd numbered elements have similar slopes and and integrated functional equations and even numbered equations have similar slopes and integrated functional equations so that equations for element 1 and 2 are employed for all computations of other elements but taking the element topology definition of Table 1 for nodes numbering into consideration. The computations of this section using equations (24), (25) and (26) for odd numbered elements and (29),(30) and (31) for even numbered elements are presented in Table 2

### **Discussion and Validation of Method and Results**

The specification of boundary conditions was based on a study of Ihueze (2005) working on compressive failure of GRP composites and on the literature that the compressive strength of GRP composites is about 50 to 60% tensile strength of GRP composites, as 154MPa. The critical or buckling strength of GRP composite evaluated by Ihueze (2005) is 24MPa, so that the functional evaluated at approximately 24MPa as presented in column 5 of Table 2 is a recognizable physical quantity, called the buckling, Strength (Ihueze 2005). Since the minimization of function,  $\frac{\partial X^e}{\partial \Phi_F}$ , at every node is greater than 0 or is positive, the optimum value of the function,  $\Phi$  is a minimum and the values of the function  $\Phi$  at the nodes are fairly constant as summarized in column 3 of Table 2. The values of  $\Phi$  are defined so that no discontinuity arises between adjacent elements. The slopes  $\partial\Phi/\partial x$  and  $\partial\Phi/\partial y$  presented in column 4 of Tables 2 on the connecting interfaces between elements are therefore finite and no contribution arises.

The polynomial model,  $P_2(x) = 23.94208 - 161.452 x + 3678.429 x^2$  representing the compressive failure response of GRP composites obtained by Ihueze(2005) was solved by gradient search method to obtain the minimum buckling strength of GRP as 24MPa (Ihueze 2006) .

This result is the same as the result obtained by solving the functional of function, X expressed in (26) and (31) and results presented in column5of Table2. The compressive strength of GRP is also about 154MPa ( Ihueze and Enetanya 2009)and this is the same as the value obtained for the function through the finite element method of this work for the region as in column3 of Table 2 .

**Conclusion**

The buckling strength is evaluated as the minimum value of a function within its domain and it represents the value of a functional of the function.The procedures of this work successfully solved compressive function represented as 2-D Laplace equation and equally can be employed when solutions of the following field function problems are needed:

- **Pure torsion of a non-homogeneous shaft expressed as**

$$\left(\frac{1}{G} \frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y} \left(\frac{1}{G} \frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial z} + 2 \theta = 0 \tag{88}$$

where  $\Phi$  = stress function  
 $G$  = shear modulus  
 $\theta$  = angle of twist per unit length of shaft

- **Flow through an anisotropic porous foundation**

$$\frac{\partial}{\partial x} (k_x \frac{\partial H}{\partial x}) + \frac{\partial}{\partial y} (k_y \frac{\partial H}{\partial y}) = 0 \tag{89}$$

where  $k_x, k_y$  = permeability coefficients in direction of the(inclined)principal axis

$H$  = flow function

- **Axi-symmetric heat flow**

$$\frac{\partial}{\partial r} (rk \frac{\partial T}{\partial r}) + \frac{\partial}{\partial z} (rk \frac{\partial T}{\partial z}) = 0 \tag{90}$$

where  $T$  = temperature  
 $k$  = conductivity  
 $r, z$  =radial and axial distances replacing x and z coordinates

○ **Hydrodynamic pressures on moving surfaces**

$$\text{Del}^2 P = \tag{91}$$

Where P = fluid pressure

○ **Time dependent field problems**

This covers problems on diffusion, vibration, creep functions etc. The governing equation is expressed as (Zienkiewicz and Cheung, 1967).

$$\left(k_x \frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial \Phi}{\partial z}\right) + Q - C \frac{\partial \Phi}{\partial t} = 0 \tag{92}$$

By applying the usual variation procedure as before the general functional which has to be for (92) minimized becomes

$$x = \iiint \left\{ \frac{1}{2} \left[ k_x \left(\frac{\partial \Phi}{\partial x}\right)^2 + k_y \left(\frac{\partial \Phi}{\partial y}\right)^2 + k_z \left(\frac{\partial \Phi}{\partial z}\right)^2 \right] - (Q - C \frac{\partial \Phi}{\partial t}) \Phi \right\} dx dy dz \tag{93}$$

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**Table 2 Post Processing of FEM results**

Element	Nodes			Function, $\Phi$			Slope		$X^c$	$\frac{\partial X^c}{\partial \Phi_i}$		
	1	2	8	154.00	154.17	154.17	$\frac{\partial \Phi}{\partial x}$	$\frac{\partial \Phi}{\partial y}$				
1	1	2	8	154.00	154.17	154.17	0.01	0.00	23.78	-0.15	0.30	0.15
2	1	8	7	154.00	154.17	154.00	0.01	0.00	23.73	0.15	0.15	0.004
3	2	3	9	154.17	154.68	154.00	0.03	0.00	24.04	-0.45	0.60	0.15
4	2	7	8	154.17	154.00	154.17	0.03	0.00	23.88	0.15	0.45	-0.30
5	3	4	10	154.68	154.68	154.68	0.00	0.00	23.93	0.00	0.15	0.15
6	3	10	9	154.68	154.68	154.68	0.00	0.00	23.93	0.15	0.00	0.15
7	4	5	11	154.68	154.17	154.17	-0.03	0.00	23.88	0.45	-0.30	0.15
8	4	10	9	154.68	154.68	154.68	-0.03	0.00	24.04	0.15	-0.45	0.60
9	5	6	12	154.17	154.00	154.00	-0.01	0.00	23.73	0.15	0.004	0.15
10	5	12	11	154.17	154.00	154.17	-0.01	0.00	23.78	0.15	-0.15	0.30