Abyss. J. Sci. Technol. Vol. 1, No. 1, 2016, 1-5

Abyssinia Journal of Science and Technology

Unified Fractional Derivative Formulae for the Generalized Mittag-Leffler Function

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ABSTRACT

The aim of this paper is to evaluate two unified fractional derivative involving the product of generalized Mittag-Leffler function and Appell function $F_3(.)$. These integrals are further applied in proving two theorems on Marichev-Saigo-Maeda fractional derivative operators. The results are expressed in terms of generalized Wright function and generalized hypergeometric functions ${}_pF_q(.)$. Further, we point out also their relevance.

Keywords: Marichev-Saigo-Maeda Fractional derivative operators, Generalized Mittag-Leffler function, Generalized Wright function, Generalized Hypergeometric series.

Mathematics Subject Classification: 26A33, 33B15, 33C05, 33C99, 44A10

INTRODUCTION

The fractional calculus operators involving various special functions have found significant importance and applications in modeling of relevant systems in various fields science and engineering, such as turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astro-physics, and in quantum mechanics. Therefore, a remarkably large number of authors have studied, in depth, the properties, applications, and different extensions of various operators of fractional calculus. For detailed account of fractional calculus operators along with their properties and applications, one may refer to the research monographs Miller & Ross (1993); Samko et al. (1993) and Kiryakova (1994).

The function $E_\alpha(z)$ introduced and defined by Mittag-Leffler (1903) as:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+1)} z^n \quad (\alpha \in \mathbb{C});\ \mathrm{Re}(\alpha) > 0. \qquad \text{(Eq. 1.1)}$$

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A further, two-index generalization of this function was given by Wiman (1905) as:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha n+\beta)} z^n \quad (\alpha, \beta \in \mathbb{C}), \qquad \text{(Eq. 1.2)}$$

where $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$.

By means of the series representation a generalization of Mittag-Leffler function (Eq. 1.2) was introduced by Prabhakar (1971) as:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)\, n!} z^n. \qquad \text{(Eq. 1.3)}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ $(\mathrm{Re}(\alpha) > 0)$. Further, it is an entire function of order $[\mathrm{Re}(\alpha)]^{-1}$ (see (Prudnikov et al. (1972), p.7)).

Shukla and Prajapati (2007), (see (Srivastava and Tomovski (2009)) defined and investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ as:

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \qquad \text{(Eq. 1.4)}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$,

$\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$, $q \in (0,1) \cup \mathbb{N}$.

Since the Mittag-Leffler function provides solutions to certain problems formulated in terms of fractional order differential, integral and difference equations, therefore, a number of useful generalization of the this function has been introduced and studied many authors. Recently, Salim and Faraj (2012) has introduced and studied a new generalization of the Mittag-Leffler function, by means of the power series:

$$E_{¬,°,p}^{¢,¶,q}(z) = \sum_{n=0}^{\infty} \frac{(¢)_{qn}}{\Gamma(¬ n+°)(¶)_{pn}} z^n, \qquad \text{(Eq. 1.5)}$$

where $¬, °, ¢, ¶ \in \mathbb{C}$; $\Re(¬), \Re(°), \Re(¢), \Re(¶)$, $p, q > 0$ such that $q \le \Re(¬) + p$.

The generalized Wright hypergeometric function ${}_p\cdot_q(z)$, for $z \in C$, complex $a_i, b_j \in \mathbb{C}$, and $~_i, \;_j \in R$ $(~_i, \;_j \neq 0;\ i = 1,2,...,p;\ j = 1,2,...,q)$ is defined as below:

$${}_p\cdot_q(z) = {}_p\cdot_q\left[\begin{matrix}(a_i, ~_i)_{1,\ p}\\ (b_j, \ _j)_{1,\ q}\end{matrix}\Big| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i + ~_i k) z^k}{\prod_{j=1}^{q}\Gamma(b_j + \ _j k)k!}. \qquad \text{(Eq. 1.6)}$$

Wright (1935) introduced the generalized Wright function (Eq. 1.6) and proved several theorems on the asymptotic expansion of ${}_p\cdot_q(z)$ (see (Wright (1935); Wright (1940 a) and Wright (1940 b)) for all values of the argument Z, under the condition:

$$\sum_{j=1}^{q} \ _j - \sum_{i=1}^{p} ~_i > -1.$$

The generalized hypergeometric function for complex $a_i, b_j \in \mathbb{C}$ and $b_j \neq 0, -1, \cdots$ $(i = 1,2,\cdots,p;$ $j = 1,2,\cdots,q)$ is given by the power series Erdélyi (1953).

$${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z) = \sum_{r=0}^{\infty} \frac{(a_1)_r\cdots(a_p)_r z^r}{(b_1)_r\cdots(b_q)_r r!}, \qquad \text{(Eq. 1.7)}$$

where for convergence, we have $|z| < 1$ if $p = q+1$ and for any z if $p \le q$. The function (Eq. 1.7) is a special case of the generalized Wright function (Eq. 1.6) for $~_1 = \cdots = ~_p = \ _1 = \cdots = \ _q = 1$:

$${}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)} {}_p\cdot_q\left[\begin{matrix}(a_i, 1)_{1,\ p}\\ (b_j, 1)_{1,\ q}\end{matrix}\Big| z\right] \qquad \text{(Eq. 1.8)}$$

A useful generalization of the hypergeometric fractional integrals, including the Saigo operators, Saigo (1978) and Saigo (1979), has been introduced by Marichev (1974) (see details in Samko et al. ((1993), p. 194, (10.47)) and later extended and studied by Saigo and Maeda ((1998), p.393, Eq. (4.12) and (4.13)), in term of any complex order with Appell function $F_3(.)$ in the kernel, as follows:

Let $~, ~', \ , \ ', ¥ \in \mathbb{C}$ and $x > 0$, then the generalized fractional calculus operators (Marichev-Saigo-Maeda operators) involving the Appell function, or Horn's F_3-function are defined by the following equations:

$$\left(I_{0+}^{~,~',\ ,\ ',¥} f\right)(x) = \frac{x^{-~}}{\Gamma(¥)}\int_0^x (x-t)^{¥-1} t^{-~'} F_3\left(~, ~', \ , \ '; ¥; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt, \quad (\Re(¥) > 0). \qquad \text{(Eq. 1.9)}$$

$$\left(I_{-}^{~,~',\ ,\ ',¥} f\right)(x) = \frac{x^{-~'}}{\Gamma(¥)}\int_x^\infty (t-x)^{¥-1} t^{-~} F_3\left(~, ~', \ , \ '; ¥; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt, \quad (\Re(¥) > 0). \qquad \text{(Eq. 1.10)}$$

$$\left(D_{0+}^{~,~',\ ,\ ',¥} f\right)(x) = \left(I_{0,+}^{-~',-~,-\ ',-\ ,-¥} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-~',-~,-\ '+k,-\ ,-¥+k} f\right)(x),$$
$$\Re(¥) > 0;\ k = [\Re(¥)] + 1. \qquad \text{(Eq.1.11)}$$

$$\left(D_{-}^{~,~',\ ,\ ',¥} f\right)(x) = \left(I_{0,-}^{-~',-~,-\ ',-\ ,-¥} f\right)(x) = \left(-\frac{d}{dx}\right)^k \left(I_{0+}^{-~',-~,-\ ',-\ +k,-¥+k} f\right)(x),$$
$$\Re(¥) > 0;\ k = [\Re(¥)] + 1. \qquad \text{(Eq. 1.12)}$$

For the definition of the Appell function $F_3(.)$ the interested reader may refer to the monograph by Srivastava and Karlsson (1985); Erdélyi et al. (1953) and Prudnikov et al. (1992). Following Saigo (1978) the image formulas for a power function, under operators (Eq. 1.9) and (Eq. 1.10), are given by:

$$\left(I_{0+}^{~,~',\ ,\ ',¥} x^{°-1}\right)(x) = x^{°-~-~'+¥-1}\,\Gamma\left[\begin{matrix}°, °+¥-~-~'-\ , °+\ '-~'\\ °+\ ', °+¥-~-~', °+¥-~'-\ \end{matrix}\right], \qquad \text{(Eq. 1.13)}$$

where $\Re(°) > max\{0, \Re(~+~'+\ -¥), \Re(~'-\ ')\}$ and $\Re(¥) > 0$.

$$\left(I_{-}^{\sim,\sim',\ ,\ ',¥}x^{°-1}\right)(x)=x^{°-\sim-\sim'+¥-1}$$
$$\times\Gamma\begin{bmatrix}1-°-\ ,1-°-¥+\sim+\sim',\ 1-°+\sim+\ '-¥\\ 1-°,1-°+\sim+\sim'+\ '-¥,\ 1-°+\sim-\end{bmatrix},$$

(Eq. 1.14)

where

$\Re(°)<1+min\{\Re(-\ ),\Re(\sim+\sim'-¥),\Re(\sim+\ '-¥)\}$

and $\Re(¥)>0$.

The symbol occurring in (Eq. 1.13) and (Eq. 1.14) is given by: $\Gamma\begin{bmatrix}a,\ b,\ c\\ d,\ e,\ f\end{bmatrix}=\dfrac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}$.

The computations of fractional integrals (and fractional derivatives) of special functions of one and more variables are important from the point of view of the usefulness of these results Purohit et al. (2011); Purohit et al. (2012) and Suthar and Purohit (2014) in the evaluation of generalized integrals and the solution of differential and integral equations. Motivated by these avenues of applications, here we establish two image formulas for the generalized Mittag-Leffler function (Eq. 1.5), involving left and right sided operators of Saigo-Meada fractional integral operators, in term of the generalized Wright function.

2. MAIN RESULTS

In this section, we establish image formulas for the generalized Mittag-Leffler function involving left and right sided operators of Saigo-Maeda fractional derivative operators (Eq. 1.11) and (Eq. 1.12); in term of the generalized Wright function. These formulas are given by the following theorems:

Theorem 2.1. Let $\sim,\sim',\ ,\ ',¥,³,°\in\mathbb{C}$ and $p,q>0,\ ¬>0,\ q\le\Re(¬)+p$ be such that $\Re(¥)>0$, $\Re(³)>-1,\ \Re(°+³)>max\,[0,\Re(¥-\sim-\sim-\sim'-\ ),\Re(\ -\sim)]$, then there hold the formula:

$$\left(D_{0+}^{\sim,\sim',\ ,\ ',¥}\left(t^{°-1}E_{¬,°,p}^{¢,¶,q}[ct^{³}]\right)\right)(x)=\frac{x^{°+\sim+\sim'-¥-1}\Gamma(¶)}{\Gamma(¢)}$$
$$\times{}_4\cdot{}_4\left[\begin{matrix}(°-¥+\sim+\sim'+\ ',³),(°-\ +\sim,³),(¢,\ q),\ (1,\ 1)\\ (°-¥+\sim+\sim',³),(°-¥+\sim+\ ',³),(°-\ ,³),(¶,\ p)\end{matrix}\,\middle|\,(cx)^{³}\right].$$

(Eq. 2.1)

Proof. On using (Eq. 1.5) and writing the function in the series form, the left hand side of (Eq. 2.1), leads to

$$\left(D_{0+}^{\sim,\sim',\ ,\ ',¥}\left(t^{°-1}E_{¬,°,p}^{¢,¶,q}[ct^{³}]\right)\right)(x)$$
$$=\sum_{n=0}^{\infty}\frac{(¢)_{qn}(c)^{³n}}{\Gamma(¬n+°)(¶)_{pn}}\left(I_{0,+}^{-\sim',-\sim,-\ ',-\ ,-¥}t^{°+³n-1}\right)(x),$$

(Eq. 2.2)

Now, upon using the image formula (Eq. 1.13), which is valid under the conditions stated with Theorem 2.1, we get

$$\left(D_{0+}^{\sim,\sim',\ ,\ ',¥}\left(t^{°-1}E_{¬,°,p}^{¢,¶,q}[ct^{³}]\right)\right)(x)=$$
$$\frac{x^{°+\sim+\sim'-¥-1}\Gamma(¶)}{\Gamma(¢)}\sum_{n=0}^{\infty}\frac{\Gamma(¢+qn)\Gamma(°-¥+\sim+\sim'+\ '+³n)}{\Gamma(¶+pn)\Gamma(°-¥+\sim+\sim'+³n)}$$
$$\times\frac{\Gamma(°-\ +\sim+³n)\Gamma(1+n)}{\Gamma(°+¥+\sim+\ '+³n)\Gamma(°-\ +³n)}\frac{((cx)^{³})^{n}}{n!},$$

(Eq. 2.3)

Interpreting the right-hand side of the above equation, in view of the definition (Eq. 1.6), we arrive at the result (Eq. 2.1).

Theorem 2.2. Let $\sim,\sim',\ ,\ ',¥,³,°,«\in\mathbb{C}$ and $¬>0$, $p,q>0,\ \ q\le\Re(¬)+p$, such that $\Re(¥)>0,\ \ \Re(³)>-1$, $\Re(°-³)<1+min\,[\Re(\ '),\ \Re(¥-\sim-\sim'),\ \Re(¥-\sim'-\ )]$, then the following formula holds true:

$$\left(D_{-}^{\sim,\sim',\ ,\ ',¥}\left(t^{-°-«}E_{¬,°,p}^{¢,¶,q}[ct^{-³}]\right)\right)(x)=\frac{x^{-°-«+\sim+\sim'-¥}\ \Gamma(¶)}{\Gamma(¢)}$$

$$\times{}_5\cdot{}_5\left[\begin{matrix}(°+«-\sim-\sim'+¥,³),\ (°+«-\sim'-\ +¥,³),\\ (°+«+\ ',³),(¢,q),\ (1,\ 1)\\ (°+«,³),\ (°+«-\sim-\sim'-\ +¥,³),\\ (°+«-\sim'+\ ',³),\ (°,³),\ (¶,\ p)\end{matrix}\,\middle|\,(cx)^{-³}\right].$$

(Eq. 2.4)

Proof. By using (Eq. 1.5), the left had side of (Eq. 2.4), can be written as:

$$\left(D_{-}^{\sim,\sim',\ ,\ ',¥}\left(t^{-°-«}E_{¬,°,p}^{¢,¶,q}[ct^{-³}]\right)\right)(x)$$
$$=\sum_{n=0}^{\infty}\frac{(¢)_{qn}(c)^{-³n}}{\Gamma(¬n+°)(¶)_{pn}}\left(I_{0+}^{-\sim',-\sim,-\ ',-\ ,-¥}t^{-°-«-³n}\right)(x),$$

(Eq. 2.5)

which on using the image formula (Eq. 1.11), arrive at

$$\left(D_{-}^{\sim,\sim',\ ,\ ',¥}\left(t^{-°-«}E_{¬,°,p}^{¢,¶,q}[ct^{-³}]\right)\right)(x)=\frac{x^{-°-«+\sim+\sim'-¥}\ \Gamma(¶)}{\Gamma(¢)}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(°+«-\sim-\sim'+¥+³n)\Gamma(°+«-\sim'-\ +¥+³n)}{\Gamma(°+«+³n)\ \Gamma(°+«-\sim-\sim'-\ +¥+³n)}$$
$$\times\frac{\Gamma(°+«-\sim'+\ '+³n)\Gamma(¢+qn)\ \Gamma(1+r)}{\Gamma(°+«-\sim'+\ '+³n)\ \Gamma(°+³n)\Gamma(¶+pn)}\frac{((cx)^{-³})^{n}}{n!}$$

(Eq. 2.6)

Interpreting the right-hand side of the above equation, in view of the definition (Eq. 1.6), we arrive at the result (Eq. 2.4).

On setting $¶=p=1$ in (Eq. 2.1) and (Eq. 2.4), we obtained the following particular case of theorems.

Corollary 2.1. Let $~,~',\ ,\ ',¥,³,° \in \mathbb{C}$ and $¬>0$, $q \le \Re(¬)+1$ be such that $\Re(¥)>0, \Re(³)>-1$, $\Re(°+³)> max\,[0, \Re(¥ -~ -~' - \ ), \Re(\ -~)]$, then there hold the formula:

$$\left(D_{0+}^{~,~',\ ,\ ',¥}\left(t^{°-1}E_{¬,°}^{¢,q}[ct^{³}]\right)\right)(x)=\frac{x^{°+~+~'-¥-1}}{\Gamma(¢)}$$

$$\times\ {}_3\Psi_3\left[\begin{matrix}(°-¥+~+~'+\ ',³),\ (°-\ +~,³),(¢,\ q)\\ (°-¥+~+~',³),\ (°-¥+~+\ ',³),(°-\ ,³)\end{matrix}\Big|(cx)^{³}\right].$$

(Eq. 2.7)

Corollary 2.2. Let $~,~',\ ,\ ',¥,³,°,« \in \mathbb{C}$ and $¬>0$, $p,\ q>0$, $q\le\Re(¬)+1$, such that $\Re(¥)>0, \Re(³)>-1$, $\Re(°-³)<1+min\,[\Re(\ '), \Re(¥-~-~'), \Re(¥-~'-\ )]$, then the following formula holds true:

$$\left(D_{-}^{~,~',\ ,\ ',¥}\left(t^{-°-«}E_{¬,°}^{¢,q}[ct^{-³}]\right)\right)(x)=\frac{x^{-°-«+~+~'-¥}}{\Gamma(¢)}$$

$$\times\ {}_4\Psi_4\left[\begin{matrix}(°+«-~-~'+¥,³),\ (°+«-~'-\ +¥,³),\\ (°+«+\ ',³),\ (¢,q)\\ (°+«,³),\ (°+«-~-~'-\ +¥,³),\\ (°+«-~'+\ ',³),\ (°,³)\end{matrix}\Big|(cx)^{-³}\right].$$

(Eq. 2.8)

3. SPECIAL CASES

In this section, we consider some special cases of the main results derived in the preceding section.

If we set $~=0$ in the operators (Eq. 1.9) and (Eq. 1.10), then we have the following known identities:

$$\left(D_{0+}^{~+\ ,0,-¦,\ ',~}f\right)(x)=\left(D_{0+}^{~,\ ,¦}f\right)(x). \quad \text{(Eq. 3.1)}$$

$$\left(D_{-}^{~+\ ,0,-¦,\ ',~}f\right)(x)=\left(D_{-}^{~,\ ,¦}f\right)(x). \quad \text{(Eq. 3.2)}$$

where the hypergeometric operators, appeared in the right hand side are due to Saigo (1978), defined as:

$$\left(I_{0+}^{~,\ ,¦}f\right)(x)=\frac{x^{-~-\ }}{\Gamma(~)}$$

$$\times\int_0^x (x-t)^{~-1}\,{}_2F_1\left(~+\ ,-¦;~;1-t/x\right)f(t)dt, \quad \text{(Eq. 3.3)}$$

$$\left(I_{-}^{~,\ ,¦}f\right)(x)=\frac{1}{\Gamma(~)}$$

$$\times\int_x^\infty (t-x)^{~-1}\,t^{-~-\ }\,{}_2F_1\left(~+\ ,-¦;~;1-x/t\right)f(t)dt.$$

(Eq.3.4)

$$\left(D_{0+}^{~,\ ,¦}f\right)(x)=\left(I_{0+}^{-~,-\ ,~+¦}f\right)(x)$$

$$=\left(\frac{d}{dx}\right)^k\left(I_{0+}^{-~+k,-\ -k,~+¦-k}f\right)(x)$$

(Eq. 3.5)

$\Re(~)>0;\quad k=[\Re(~)]+1.$

$$\left(D_{-}^{~,\ ,¦}f\right)(x)=\left(I_{-}^{-~,-\ ,~+¦}f\right)(x)$$

$$=\left(-\frac{d}{dx}\right)^k\left(I_{-}^{-~+k,-\ -k,~+¦}f\right)(x)$$

(Eq. 3.6)

$\Re(~)>0;\quad k=[\Re(~)]+1.$

Therefore, if we set $~'=0,\ \ =-¦,\ ¥=~$ and replace $~$ by $~+\ $ in (Eq. 2.1) and (Eq. 2.2), we get the following results, involving the left and right hand sided Saigo type derivative operators:

Corollary 3.1 Let $~,\ \ ,¦,¢,¶,° \in \mathbb{C}$ and $p,q,¬>0$, $q\le\Re(¬)+p$, $\Re(-~)>0$, $\Re(°+~+¦+\ )>0$, then there hold the formula:

$$\left(D_{0+}^{~,\ ,¦}\left(t^{°-1}E_{¬,°,p}^{¢,¶,q}[ct^{³}]\right)\right)(x)=\frac{x^{°+\ -1}\Gamma(¶)}{\Gamma(¢)}$$

$$\times\ {}_3\Psi_3\left[\begin{matrix}(°+~+¦+\ ,³),\ (¢,\ q),\ (1,\ 1)\\ (°+\ ,³),\ (°+¦,³),\ (¶,\ p)\end{matrix}\Big|(cx)^{³}\right]. \quad \text{(Eq. 3.7)}$$

Corollary 3.2 Let $~,\ \ ,¦,¢,¶,°,« \in \mathbb{C}$ and $p,q,¬>0$, $q\le\Re(¬)+p$, $\Re(°-~)>\max[\Re(\ ),\Re(~+¦)]$, then the following formula hold:

$$\left(D_{-}^{~,\ ,¦}\left(t^{-«-°}E_{¬,°,p}^{¢,¶,q}[ct^{-³}]\right)\right)(x)=\frac{x^{-°-«+\ }\Gamma(¶)}{\Gamma(¢)}$$

$$\times\ {}_4\Psi_4\left[\begin{matrix}(°+«-\ ,³),\ (°+«+~+¦,³),(¢,\ q),\ (1,1)\\ (°,³),\ (°+«,³),\ (°+«-\ +¦,³),(¶,p)\end{matrix}\Big|(cx)^{-³}\right].$$

(Eq. 3.8)

On setting $¶=p=1$ in (Eq. 3.5) and (Eq. 3.6), we obtained the following particular case of corollaries.

Corollary 3.3 Let $~,\ \ ,¦,¢,¶,° \in \mathbb{C}$ and $p,q,¬>0$, $q\le\Re(¬)+1$, $\Re(-~)>0$, $\Re(°+~+¦+\ )>0$, then there hold the formula:

$$\left(D_{0+}^{~,\ ,¦}\left(t^{°-1}E_{¬,°}^{¢,q}[ct^{³}]\right)\right)(x)=\frac{x^{°+\ -1}}{\Gamma(¢)}$$

$$\times\ {}_2\Psi_2\left[\begin{matrix}(°+~+¦+\ ,³),\ (¢,\ q)\\ (°+\ ,³),\ (°+¦,³)\end{matrix}\Big|(cx)^{³}\right]. \quad \text{(Eq. 3.9)}$$

Corollary 3.4 Let $~,\ \ ,¦,¢,¶,°,« \in \mathbb{C}$ and $¬>0$, $p,q>0$, $q\le\Re(¬)+1$, $\Re(°-~)>\max[\Re(\ ),\Re(~+¦)]$ then the following formula hold:

$$\left(D_{-}^{~,\ ,¦}\left(t^{-«-°}E_{¬,°}^{¢,q}[ct^{-³}]\right)\right)(x)=\frac{x^{-°-«+\ }}{\Gamma(¢)}$$

$$\times\ {}_3\Psi_3\left[\begin{matrix}(°+«-\ ,³),\ (°+«+~+¦,³),(¢,\ q)\\ (°,³),\ (°+«,³),\ (°+«-\ +¦,³)\end{matrix}\Big|(cx)^{-³}\right].$$

(Eq. 3.10)

Remark 1. If we set $q=1$ in corollary 3.3, we arrive at the known result given by Chaurasia and Pandey ((2010), Eq. 5.1).

Remark 2. If we set $q=1$, « = – ~ in corollary 3.4, we arrive at the known result given by Chaurasia and Pandey ((2010), Eq. 6.1).

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