

GENERALISED DELETION DESIGNS

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ABSTRACT:- *In this paper asymmetrical single replicate factorial designs are constructed from symmetrical single replicate factorial designs using the deletion technique. The study is along the lines of Voss(1986), Chauhan(1989) and Gachii and Odhiambo(1997). We give results for the general order deletion designs of the form $s^{n-m_1}(s-L)^{m_1}$ which are proper, for $1 \leq L \leq s-1$ and m_1 less than or equal to the number of generators of the preliminary single replicate generalized cyclic design. We generalize results by earlier authors. Results identifying the set of estimable factorial effects of the deletion designs based on the information available from the preliminary factorial design are given. Simple formulas for calculating the loss of information due to confounding with blocks are given. Efficiency with respect to the number of treatment combinations needed to estimate a given set of interactions of the preliminary factorial design is compared with that of the resulting deletion designs.*

Key words and phrases: Factorial spaces, generalized cyclic designs, loss of information.

INTRODUCTION

To construct a single replicate factorial design having n factors F_1, F_2, \dots, F_n , factor F_i occurring at s_i levels, we first construct a single replicate preliminary block design, say d_p , in n factors, factor F_i occurring at r_i levels, using any of the known methods such that $r_i \geq s_i$ for $i = 1, 2, \dots, n$. We can then select $l_i = r_i - s_i$ levels of the i -th factor of d_p and delete from d_p all treatment combinations where factor F_i occurs at any of the l_i selected levels. If levels are deleted from k factors we refer to the resulting design as a k -th order deletion design.

Earlier work on this area was done by Bose (1947). He used finite Euclidean geometry to construct asymmetrical factorial designs in blocks. Kishen and Srivastava extended the method of finite geometries to the construction of balanced confounded asymmetrical factorial designs thereby introducing the idea of deletion.

John and Dean (1975) proposed a simple method of confounding based on generalized cyclic designs from a

set of generating treatments or generators and showed that the confounding patterns could easily be determined from these generators.

Voss (1986) constructed nearly orthogonal single replicate factorial designs in blocks. He uses the deletion technique, where he deletes from the first factor without loss of generality to obtain first order deletion designs. Chauhan (1989) generalized the work by Voss (1986), by constructing efficient single replicate designs using the deletion technique. Starting from an s^n generalized cyclic design, she constructed efficient proper single replicate deletion designs of the form $(s-L)s^{n-1}$. Gachii and Odhiambo (1998) Constructed deletion designs of the form $s^{n-m_1}(s-L)^{m_1}$ which are proper for $1 \leq L \leq s-1$ and m_1 less than or equal to the number of generators of the preliminary single replicate generalized cyclic design.

The objective of the present paper is to give general results useful in evaluating the efficiency of the constructed deletion designs based on the loss of information on the estimable effects.

To construct the preliminary single replicate factorial design in which factor F_i occurs at $r_i = s$ levels for $i = 1, 2, \dots, n$ we use the method by John and Dean (1975). We derive expressions for loss of information due to deletion on the estimable effects. Confounding patterns on the deletion designs are also given.

NOTATION

We assume the fixed effects linear model

$$y_{ah} = \mu + \tau_a + \beta_h + \epsilon_{ah} \tag{2.1}$$

where y_{ah} denotes the observed yield from treatment combination \mathbf{a} in the h -th block; τ_a denotes the effect of treatment combination \mathbf{a} ; β_h denotes the effect of the h -th block and ϵ_{ah} are uncorrelated random errors with zero mean and variance σ^2 . Let $\mathbf{y} = (y_{ah})$ and $\boldsymbol{\tau} = (\tau_a)$ denote the $v \times 1$ vectors of observations and treatment effects respectively each lexicographically ordered by \mathbf{a} . That is a treatment combination $\mathbf{a} = a_1 a_2 \dots a_n$ appears before another treatment combination $\mathbf{a}^* = a_1^* a_2^* \dots a_n^*$ if and only if for the first u such that $a_u \neq a_u^*$ we have $a_u < a_u^*$ for $1 \leq u \leq n$. The i -th row of \mathbf{y} and $\boldsymbol{\tau}$ above corresponds to the i -th treatment combination in the above arrangements of the v treatment combinations.

We shall denote the incidence matrix, the intra block matrix, the diagonal matrix of block sizes and the number of blocks respectively by N, A, K and b . The i -th row of the incidence matrix N corresponds to the i -th lexicographically ordered treatment combination \mathbf{a} . the $q \times 1$ vectors of ones and of zeros will be denoted by $\mathbf{1}_q$ and $\mathbf{0}_q$ respectively.

A generalized interaction will be denoted by α^x where $x = x_1 x_2 \dots x_n$ such that $x_j = 1$ if F_j is present in the interaction and $x_j = 0$ otherwise. A $v \times 1$ contrast vector will be denoted by \mathbf{c}^x where

$$\mathbf{c}^x = \mathbf{c}_1^{x_1} \otimes \mathbf{c}_2^{x_2} \otimes \dots \otimes \mathbf{c}_n^{x_n}$$

with $\mathbf{c}_j^{x_j}$ being an $S_j \times 1$ vector of ones if $x_j = 0$,

otherwise it is an $S_j \times 1$ contrast vector. The minimum variance unbiased estimator of the generalised interaction α^x is represented by p while those corresponding to the deletion design will carry no subscript.

LOSS OF INFORMATION ON ESTIMABLE EFFECTS

We shall use the notation

$$\mathbf{d}_{a_j} = \mathbf{1}'_{(s-1)} D_j \mathbf{p}^{a_j} D_j' \mathbf{1}_{(s-1)} \tag{3.1}$$

where D_j is the matrix obtained from an $s \times s$ identity matrix by deleting the t -th row if the t -th level is deleted from factor F_j in the preliminary design d_p to obtain d and $s_j \times s_j$ is an $s \times s$ permutation matrix with 1 in the a_j -th column of the o -th row.

We shall also write

$$\mathbf{d}_{a_j}^* = \mathbf{c}'_j D_j \mathbf{P}^{a_j} D_j' \mathbf{c}_j \tag{3.2}$$

where \mathbf{c}_j is a contrast vector from the column space of the matrix $\mathbf{S}\mathbf{I}_{(s)} - \mathbf{J}_{(s)}$. Again where $\mathbf{I}_{(s)}$ is an s -dimensional identity matrix and $\mathbf{J}_{(s)} = \mathbf{1}_{(s)} \mathbf{1}'_{(s)}$.

Following Dean (1978), for a given contrast vector \mathbf{c}^x , the loss of information ψ_x , $0 \leq \psi_x \leq 1$, due to confounding with blocks, is given by

$$\psi_x = \frac{\mathbf{c}^{x'} \mathbf{N} \mathbf{K}^{-1} \mathbf{N}' \mathbf{c}^x}{\mathbf{c}^{x'} \mathbf{c}^x} \tag{3.3}$$

Where N is the incidence matrix and K is the diagonal matrix of block sizes.

We consider deletion designs of the form $s^{n-m} (s-1)^m$ with $b = \lambda s^{n-m}$ blocks each of size $K = (\frac{1}{\lambda}) s^{m-m} (s-1)^m$ derived from an S^n generalised cyclic design d_p with $k_p = (\frac{1}{\lambda}) s^n$ and $b_p = \lambda s^{n-m}$. Where n is the number of factors, m is the number of generators, m_1 is the order of the deletion design and $\lambda = \prod_{i=1}^m 1/b_i$ with $b_i = \text{HCF}(s, g_i)$, $i = 1, 2, \dots, m$ as given by John and Dean (1975).

We now give general results on loss of information for a given interaction effect on these designs.

Lemma 3.1:

Loss of information due to confounding with blocks on any r -factor interaction of the form $F_{i_1} F_{i_2} \dots F_{i_r}$, $\{i_1, i_2, \dots, i_r\} \subset \{1, 2, \dots, n - m_1\}$ whose levels were not deleted for all the factors in the interaction to obtain d is given by

$$\Psi_x = \frac{\lambda \sum_{a_1} \dots \sum_{a_r} w_{a_1, a_2, \dots, a_r} d_{a_1, a_2, \dots, a_r}^* d_{a_1}^* d_{a_2}^* \dots d_{a_r}^*}{(s-1)^r (s-1)^{2m_1} s^{n+m-2m_1}}$$

where

$$w_{a_1, a_2, \dots, a_r} = \begin{cases} 1, & \text{if } a = a_1 a_2 \dots a_n \text{ is in the initial block of } d_p \\ 0, & \text{otherwise} \end{cases}$$

again where $d_{a_1, a_2, \dots, a_r} = d_{a_1} \times d_{a_2} \times \dots \times d_{a_r}$ is such that it does not contain $d_{a_1}, d_{a_2}, \dots, d_{a_r}$ since for these factors we have instead $d_{a_1}^*, d_{a_2}^*, \dots, d_{a_r}^*$ which are contrast vectors from $SI_{(s)} - J_{(s)}$.

Proof: The contrast vector c^x is as given in (2.2) with c_t^x being an $s \times 1$ unit vector for t is an $(s-1) \times 1$ unit vector for $t \pm i_1, i_2, \dots, i_r, t = 1, 2, \dots, n - m_1, c_t^x$ is any of the columns of the matrix $sI_{(s)} - J_{(s)}$ for $t = i_1, i_2, \dots, i_r$. Therefore

$$c^x c^x = (s-1)^r (s-1)^{m_1} s^{n-m_1} \tag{3.4}$$

But

$$K^{-1} = (\lambda \setminus s^{m-m_1} (s-1)^{m_1}) I_{(s)^{n-m_1}} \tag{3.5}$$

and

$$c^x N N^x c^x = \sum_{a_1} \dots \sum_{a_r} w_{a_1, a_2, \dots, a_r} d_{a_1, a_2, \dots, a_r}^* d_{a_1}^* d_{a_2}^* \dots d_{a_r}^* \tag{3.6}$$

using (3.1) and (3.2).

Hence using (3.4), (3.5) and (3.6) we obtain Lemma 3.1.

Lemma 3.2: Loss of information due to confounding with blocks on any r -factor interaction of the form $F_{i_1}, F_{i_2}, \dots, F_{i_r}, \{i_1, i_2, \dots, i_r\} \subset \{1, 2, \dots, n - m_1\}$ whose levels were not deleted and $\{i_{r+1}, i_{r+2}, \dots, i_r\} \subset \{n - m_1 + 1, n - m_1 + 2, \dots, n\}$ whose levels were deleted to obtain d is given by

$$\Psi_x = \frac{\lambda \sum_{a_1} \dots \sum_{a_r} w_{a_1, a_2, \dots, a_r} d_{a_1, a_2, \dots, a_r}^* d_{a_1}^* d_{a_2}^* \dots d_{a_r}^*}{(s-1)^r (s-1)^{r-r} (s-1)^{2m_1} s^{n+m-2m_1}}$$

where w_{a_1, a_2, \dots, a_r} and d_{a_1, a_2, \dots, a_r} is as given in lemma 3.1 and $d_{a_1}^*, d_{a_2}^*, \dots, d_{a_r}^*$ are contrast vectors from $SI_{(s)} - J_{(s)}$ and $d_{a_{i_1+1}}^*, \dots, d_{a_{i_r}}^*$ are contrast vectors from $(s-1)I_{(s-1)} - J_{(s-1)}$.

Proof: The contrast vector c^x is as given in (2.2) with c_t^x being an $s \times 1$ unit vector for t is an $(s-1) \times 1$ unit vector for $t = n - m_1 + 1, n - m_1 + 2, \dots, n$ and $t \pm i_1, i_2, \dots, i_r$ and c_t^x

Is any of the columns of the matrix

$sI_{(s)} - J_{(s)}$ for $t = i_1, i_2, \dots, i_r, i_j \leq n - m_1, j = 1, 2, \dots, r_1$ and c_t^x is any of the columns of the matrix $(s-1)I_{(s-1)} - J_{(s-1)}$ for $t = i_{r+1}, i_{r+2}, \dots, i_r, i_j > n - m_1, j = r_1 + 1, \dots, r$. Therefore

$$c^x c^x = (s-1)^r (s-1)^{r-r} (s-1)^{m_1} s^{n-m_1} \tag{3.7}$$

Hence using (3.5), (3.6) and (3.7) we obtain Lemma 3.2

Lemma 3.3

Loss of information due to confounding with blocks on any r -factor interaction of form $F_{i_1}, F_{i_2}, \dots, F_{i_r}$ for which levels of all the factors were deleted from d_p to obtain d is given by

$$\Psi_x = \frac{\lambda \sum_{a_1} \dots \sum_{a_r} w_{a_1, a_2, \dots, a_r} d_{a_1, a_2, \dots, a_r}^* d_{a_1}^* \dots d_{a_r}^*}{(s-1)^r (s-1)^{2m_1} s^{n+m-2m_1}}$$

where w_{a_1, a_2, \dots, a_r} and d_{a_1, a_2, \dots, a_r} is as given in lemma 3.1 and $d_{a_{i_1+1}}^*, \dots, d_{a_{i_r}}^*$ are contrast vectors from $(s-1)I_{(s-1)} - J_{(s-1)}$.

Proof: The contrast vector c^x is as given in (2.2) with c_t^x being an $s \times 1$ unit vector for $t = 1, 2, \dots, n - m_1, c_t^x$ is an $(s-1) \times 1$ unit vector for $t = n - m_1 + 1, n - m_1 + 2, \dots, n$ and $t \pm i_1, i_2, \dots, i_r$ and c_t^x

Is any of the columns of the matrix $(s-1)I_{(s-1)} - J_{(s-1)}$ for $t = i_1, i_2, \dots, i_r$. Hence

$$c^x c^x = (s-1)^r (s-1)^{m_1} s^{n-m_1} \tag{3.8}$$

Hence using (3.5), (3.6) and (3.8) we obtain Lemma 3.3

CONFOUNDING IN DELETION DESIGNS

The following results in confounding in generalised cyclic designs is due to John and Dean (1975). The number of degrees of freedom confounded with blocks for any given interaction α^x is given by

$$Y^x = \frac{1}{k} \sum_{a_1} \sum_{a_2} \dots \sum_{a_n} w_{a_1, \dots, a_n} \left(\prod_{j=1}^n z_{a_j}^{x_j} \right) \quad (4.1)$$

$$\text{where } z_{a_j}^{x_j} = \begin{cases} s-1, & \text{if } a_j = 0 \quad \text{and } x_j = 1 \\ -1, & \text{if } a_j \neq 0 \quad \text{and } x_j = 1 \\ 1, & \text{if } x_j = 0 \end{cases}$$

and w_{a_1, \dots, a_n} is as given in Lemma 3.1.

If the number of degrees of freedom in (4.1) is zero, then the interaction is unconfounded with blocks and if it is

$\prod_{j=1}^n (s-1)^{x_j}$ the interaction is totally confounded with blocks. Consider any interactions between the n factors, say the interactions of the factors F_1, F_2, \dots, F_r

Let

$$A = ((a_{ij})) \quad (4.2)$$

where a_{ij} is from the i -th generator $g_i = a_{i1}, a_{i2}, \dots, a_{in}$; $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Consider all the $f \times f$ submatrices contained in the j_1 -th, j_2 -th, ..., j_r -th rows of A and let h_{j_1, \dots, j_r} be the absolute values of their determinants ($f \leq r, f \leq m$). Define as follows

$$H_f = \begin{cases} 1, & \text{if } f = 0 \\ \text{HCF}(h_{j_1, \dots, j_r} \setminus \{j_1, j_2, \dots, j_r\} \subset \{i_1, i_2, \dots, i_r\}) & \text{if } 0 < f < m \\ 0, & \text{if } f > m \end{cases} \quad (4.3)$$

In our case the treatment combinations in the initial block are of the form

$$u_1 g_1 + u_2 g_2 + \dots + u_m g_m \quad (u_i = 0, 1, \dots, k_i - 1; i = 1, 2, \dots, m)$$

The number of treatments in the initial block with i_1 -th, i_2 -th, ..., i_r -th factors all zero is given by $s^{m-r} w_{i_1, \dots, i_r}$ where

$$w_{i_1, \dots, i_r} = \begin{cases} \prod_{f=1}^r \text{HCF}(s, H_f / H_{f-1}), & \text{if } r \leq m \text{ and } H_r \neq 0 \\ s^{r-g} \text{HCF}(w_{j_1, \dots, j_g} \setminus \{j_1, j_2, \dots, j_g\} \subset \{i_1, i_2, \dots, i_r\}) & \text{if } g < r \leq m \\ s^{r-g} \text{HCF}(w_{j_1, \dots, j_m} \setminus \{j_1, j_2, \dots, j_m\} \subset \{i_1, i_2, \dots, i_r\}) & \text{if } r > m \end{cases} \quad (4.4)$$

where g is such that $H_g \neq 0$ and $H_{g+1} = H_{g+2} = \dots = 0$ and

$g = 1$ if $H_1 = 0$. Let Y^x be denoted by Y_{j_1, j_2, \dots, j_h} where x has the j_1 -th, j_2 -th, ..., j_h -th digits unity and the remainder zero. Then it can be shown that for the interactions of the factors F_1, F_2, \dots, F_r , the number of degrees of freedom confounded with blocks is given by

$$Y_{i_1, i_2, \dots, i_r} = w_{i_1, i_2, \dots, i_r} - \sum_{g=1}^{r-1} (Y_{j_1, j_2, \dots, j_g} \setminus \{j_1, j_2, \dots, j_g\} \subset \{i_1, i_2, \dots, i_r\}) - 1 \quad (4.5)$$

We now give the following results on confounding in deletion designs.

Chauhan (1989) studied confounding in deletion designs and the following discussion and theorem 1 below is due to her. Let α^x be a given interaction. Then the factors F_1, F_2, \dots, F_r or simply $\{1, 2, \dots, n\}$ can be partitioned into three mutually exclusive and exhaustive sets Ω_1, Ω_2 , and Ω_3 as follows: Ω_1 contains the factors whose levels were not deleted from d_p to obtain d , that is the factors $\{1, 2, \dots, n - m_1\}$; Ω_2 contains the factors whose levels were deleted from d_p to obtain d and these factors are not in the factorial space V_x , that is the factors $\{n - m_1 + 1, n - m_1 + 2, \dots, n - m_1 + a\}$; Ω_3 contains the factors whose levels were deleted from d_p to obtain d and these factors are in the factorial space V_x , that is the factors $\{n - m_1 + a + 1, n - m_1 + a + 2, \dots, n\}$; $a = 0, 1, 2, \dots, m_1$.

We shall write the factorial space V_x as $V(j_1, j_2, \dots, j_r)$ if $x_{j_1} = x_{j_2} = \dots = x_{j_r} = 1$ and all other x_j 's are zero, where $\{i_1, i_2, \dots, i_r\} \subset \{1, 2, \dots, n - m_1\}$, then we have the following theorem.

Theorem 1: (Chauhan (1989))

Let the contrast vector

$$c^x \in v(i_1, i_2, \dots, i_r, n - m_1 + a + 1, n - m_1 + a + 2, \dots, n)$$

and let $c_p^x = D'c^x$

then

$$c_p^x \in \oplus v_p(i_1, i_2, \dots, i_r, g, n - m_1 + a + 1, n - m_1 + a + 2, \dots, n)$$

where $g \in p(\Omega_2)$; that is g belongs to the power set of Ω_2 and \oplus denotes the direct sum. Where

$$D = D_1 \otimes D_2 \otimes \dots \otimes D_n$$

again D_j where is as defined in section 3.

The following results on deletion designs follow:

Theorem 2

If S^n is a generalised cyclic design generated by m generators such that $g_1 =$ twice the $(n-m+1)$ -th row of an identity matrix of order n and g_2, g_3, \dots, g_m are the last $(m-1)$ rows of an identity matrix of order n and if $s > 2$ is even then all the main effects and all the interactions of the first $(n-m)$ factors are totally confounded with blocks while the main effect of factor F_{n-m+1} and any effects involving this factor and any number of the first $(n-m)$ factors are partially confounded with blocks provided $n \geq 2m - 1$.

Proof: For the main effects and interactions of the first factors, we have for the main effect of factor $F_{j_1}, j_1 = 1, 2, \dots, n - m, h_{j_1} = HCF(0, 0, \dots, 0) = 0$, which gives

$$w_{j_1} = HCF(s, 0) = s$$

and hence $Y_{j_1} = s - 1 \neq 0$ since $s > 1$.

Therefore all the main effects of the first $(n - m)$ factors are totally confounded with blocks using (4.5).

For two factors interactions of factors F_{j_1} and $F_{j_2}, j_1 < j_2 = 1, 2, \dots, n - m$, the matrix A in (4.2) is the zero matrix which gives $H_2 = 0$. Therefore

$$w_{j_1 j_2} = s^{2-1} HCF(s, s) = s^2$$

$$Y_{j_1 j_2} = s^2 - 2(s-1) - 1$$

and so

$$= (s-1)^2 \neq 0$$

since $s > 1$. Hence by (4.5) all the two factors interactions of this form are totally confounded with blocks.

To obtain the result for the general case we proceed by induction. Assume the k -factor interactions are totally confounded with blocks for $k = 1, 2, \dots, n - m - 1$, that is

$$Y_{j_1 j_2 \dots j_k} = (s-1)^k \neq 0$$

We wish to show that the $(k + 1)$ - factor interactions are totally confounded with blocks. From (4.5)

$$Y_{j_1 j_2 \dots j_k j_{k+1}} = s^{k+1} - \sum_{g=1}^k (Y_{i_1 \dots i_g} \setminus \{i_1, i_2, \dots, i_g\} C \{j_1, j_2, \dots, j_{k+1}\}) - 1$$

$$= s^{k+1} - \binom{k+1}{k} Y_{j_1 j_2 \dots j_k} - \binom{k+1}{k-1} Y_{j_1 j_2 \dots j_{k-1}} - \binom{k+1}{1} Y_{j_k} - \binom{k+1}{0}$$

$$= \binom{k+1}{k+1} s^{k+1} - \binom{k+1}{k} (s-1)^k - \binom{k+1}{k-1} (s-1)^{k-1} - \binom{k+1}{1} (s-1) - \binom{k+1}{0} = (s-1)^{k+1}$$

therefore all the k -factor interactions, $k=1, 2, \dots, n-m-1$ are totally confounded with blocks.

For the main effect of factor F_{n-m+1} we have

$$h_{n-m+1} = HCF(0, 0, \dots, 0, 2, 0, \dots, 0) = 2$$

Thus

$$w_{n-m+1} = HCF(s, 2) = 2$$

and therefore

$$Y_{n-m+1} = 2 - 1 = 1 < (s-1)$$

Hence one degree of freedom from the main effect of factor F_{n-m+1} is confounded with blocks. Therefore the main effect of factor F_{n-m+1} is partially confounded.

For the two factor interactions of the form $F_{j_1} F_{n-m+1}, j_1 = 1, 2, \dots, n - m$, the matrix A is of the form

$$A = ((a_i)) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 2 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Hence $H_2 = 0$. and

$$w_{j_1 \dots j_{n-m+1}} = s^{2-1} HCF(s, 2) = 2s$$

Therefore

$$Y_{j_1 \dots j_{n-m+1}} = 2s - (s-1) - 1 - 1 \\ = s - 1 < (s-1)^2$$

and it follows that the two factor interactions of this form are partially confounded with blocks.

In general for the k -factor interaction $F_{j_1} F_{j_2} \dots F_{j_{k-1}} F_{n-m+1}$

for $j_1 < j_2 < \dots < j_{k-1} = 1, 2, \dots, n-m$, we get

$$w_{j_1 j_2 \dots j_{j-1} n-m+1} = 2s^{k-1}$$

Therefore using (4.5)

$$Y_{j_1 \dots j_{n-m+1}} = 2s^{k-1} - \left[\binom{k}{k-1} - \binom{k-1}{k-1} \right] (s-1)^{k-2} - \left[\binom{k}{k-2} - \binom{k-1}{k-2} \right] (s-1)^{k-3} - \dots \\ - \left[\binom{k}{1} - \binom{k-1}{1} \right] (s-1)^0 = (s-1)^{k-1} < (s-1)^k$$

hence all interactions involving factor F_{n-m+1} and any number of the first $(n-m)$ factors are partially confounded with blocks.

It can be shown that the $2s^{n-m} - 1$ degrees of freedom confounded with blocks are from the main effects and interactions of the first $(n-m)$ factors and from the main effect of factor and all the interactions involving factor F_{n-m+1} and any number of the first $(n-m)$ factors provided $n \leq 2m - 1$. Again the loss of information on the main effect of factor F_{n-m+1} or any two factor interactions of the form $F_j F_{n-m+1}$, $j = 1, 2, \dots, n-m$ due to confounding with blocks, is given by $\psi_x = 1/(s-1)$.

The following results are on the efficiency of the class

of deletion designs derived from the generalised cyclic designs above with respect to estimable effects.

Theorem 3: The main effect of factor F_{n-m+1} and all the interactions involving this factor and any number of the first $(n-m)$ factors are partially confounded in these deletion designs.

Proof: In this case $c^x \in V(i_1, i_2, \dots, i_e, n-m+1)$ where $\{i_1, i_2, \dots, i_e\} \subset \{1, 2, \dots, n-m\}$ for $e = 1, 2, \dots, n-m$. Hence $\Omega_1 = \{i_1, i_2, \dots, i_e\}$, $\Omega_2 = \{n-m_1+1, n-m_1+2, \dots, n\}$ and $\Omega_3 = \{\emptyset\}$. Therefore the power set Ω_2 is given by $P(\Omega_2) = \{\emptyset, \{n-m_1+1\}, \dots, \{n\}, \{n-m_1+1, n-m_1+2\}, \dots, \{n-m_1+1, n-m_1+2, \dots, n\}\}$

it follows from theorem 1 that if

$$c_p^x = D' c^x$$

then

$$c_p^x \in V_p(i_1, i_2, \dots, i_e, n-m+1) \oplus V_p(i_1, i_2, \dots, i_e, n-m+1, n-m_1+1) \oplus \dots \oplus V_p(i_1, i_2, \dots, i_e, n) \oplus \dots \oplus V_p(i_1, i_2, \dots, i_e, n-m+1, n-m_1+1, \dots, n)$$

But all these factorial spaces are partially confounded in d_p and hence the result.

Using a similar argument it can be shown that all the other main effects and interactions are fully estimable in these designs.

These deletion designs are derived from an generalised s^n cyclic design generated by m generators such that $g_1 =$ twice the $(n-m+1)$ -th row of an identity matrix of order n and g_2, g_3, \dots, g_m hence it follows that $(m-1)$ the loss of information on factor F_{n-m+1} due to confounding with blocks is given by

$$\psi_x = \frac{2 \sum_{a_{n-m+1}} \dots \sum_{a_n} w_{a_1, a_2, \dots, a_n} d_{a_1, a_2, \dots, a_n} d_{a_{n-m+1}}^*}{(s-1)(s-1)^{2m_1} s^{n+m-2m_1}}$$

using lemma 3.1 But if $a = a_1 a_2 \dots a_n$ is a treatment combination in the initial block, then

$$a_i = \begin{cases} 0, & \text{for } i = 1, 2, \dots, n - m \\ a_j, & \text{for } i = n - m + 1, n - m + 2, \dots, n \end{cases}$$

where $a_{n-m+1} = 0, 2, 4, \dots, s-2$ and $a_j = 0, 1, 2, \dots, s-1$ for $j = n - m + 2, n - m + 3, \dots, n$. Therefore for these deletion designs.

$$\Psi_x = \frac{2s^{n-m} \sum_{a_{n-m+1}} \dots \sum_{a_n} w_{00} \dots a_{n-m+1} \dots a_n d_{a_{n-m+2}} \dots a_n d_{a_{n-m+1}}^*}{(s-1)(s-1)^{2m} s^{n+m-2m}}$$

It can be shown that

$$d_{a_i} = \begin{cases} s-1, & \text{if } a_j = 0 \text{ and } l = 1, 2, \dots, s-1 \\ s-2, & \text{if } a_j \neq 0 \text{ and } l = 1 \end{cases}$$

and

$$d_{a_{n-m+1}}^* = \begin{cases} s(s-1), & \text{if } a_{n-m+1} = 0 \\ -s, & \text{if } a_{n-m+1} \neq 0 \end{cases}$$

The following result on loss of information follow.

Theorem 4: Loss of information on the main effect of factor F_{n-m+1} or any two factor interactions of the form $F_j F_{n-m+1}$, $j = 1, 2, \dots, n - m$ due to confounding with blocks, provided

$l = 1$, is given by

$$\Psi_x = \frac{1}{(s-1)^{2m+1} s^{n-m-1}} \left\{ \binom{m-1}{m-1} (s-1)^m + \binom{m-1}{m-2} (s-1)^{m-1} (s-2)^1 + \binom{m-1}{m-3} (s-1)^{m-2} (s-2)^2 + \dots + \binom{m-1}{m-(m_1+2)} (s-1)(s-2)^{m_1} + \dots + \binom{m-1}{0} (s-1)^{m-m_1} (s-2)^{m_1} \right\}$$

Example Consider a 4^4 generalised cyclic design generated by $g_1 = 0 2 0 0$, $g_2 = 0 0 1 0$ and $g_3 = 0 0 0 1$. Thus we have $b_p = 8$ blocks of size $k_p = 2 \times 4^3$. It follows by Theorem 2 that the main effect of factor F_1 is totally confounded with blocks while the main effect of factor F_2 and the two factor interaction $F_1 F_2$ are partially confounded with blocks. All the other main effects and interaction are fully estimable. Loss of information on main effects of factor F_2 or on two factor interactions $F_1 F_2$ is given by

$$\Psi_x = \frac{1}{(s-1)} = 0.33$$

For the deletion designs of the form $4^{4-m_1} (4-1)^{m_1}$, $m_1 = 1, 2$, the main effects of factor F_2 and the two factor interaction $F_1 F_2$ remain partially confounded using Theorem 3. All other main effects and intersections are fully estimable.

The loss of information on the main effect of factor or on the two factor interaction $F_1 F_2$ is now as follows;

(i) For the $4^3 \times 3$ deletion design

$$\Psi_x = \frac{1}{(4-1)^3 4} \left\{ \binom{3-1}{3-1} (4-1) + \binom{3-1}{3-2} (4-2)^1 + \binom{3-1}{0} 3 \times 2^2 \right\} = 0.213$$

(ii) For the $4^2 \times 3^2$ deletion design $\Psi_x = 0.201$

Note that for a 4^6 generalised cyclic design generated by $g_1 = 0 2 0 0 0 0$, $g_2 = 0 0 1 0 0 0$, $g_3 = 0 0 0 1 0 0$, $g_4 = 0 0 0 0 1 0$ and $g_5 = 0 0 0 0 0 1$. Thus $k_p = 2 \times 4^4$ and $b_p = 8$. The same effects that were not estimable in the 4^4 generalised cyclic design are still not estimable in this design and are the only ones. Loss of information on F_2 or the two factor interaction $F_1 F_2$ remain the same. However

(i) For the $4^5 \times 3$ deletion design $\Psi_x = 0.198$

(ii) For the deletion design $\Psi_x = 0.125$

(iii) For the $4^3 \times 3^3$ deletion design $\Psi_x = 0.103$

(iv) For the $4^2 \times 3^4$ deletion design $\Psi_x = 0.122$

CONCLUDING REMARKS

These deletion designs are sometimes more efficient than the preliminary generalized cyclic design they were derived from with respect to loss of information. For example for $l=1$ using Theorem 4 above

$$\Psi_x < \frac{(s-1)^{m-1}}{(s-1)^{2m+1} s^{m-m-1}} = s^{-(m+1)} < (s-1)^{-1}$$

For some deletion designs the effects that were non-estimable in the preliminary generalized cyclic design are the same effects that are non-estimable in the resulting deletion designs as is the case in the above example. Lastly

the deletion designs are more economical than the preliminary generalised cyclic design they are derived from because they require a smaller number of treatment combinations thus smaller number of experimental units.

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